About relations (Version 7 - final)

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Abstract

This provides some general considerations about relations in order to prepare their usage for the mathematical modelling of IES/FSTP

1 Relations

R1.1 Definition: A *relation* Rel from the set A to the set B is given by a subset $Rel \subseteq A \times B$. We also denote this by $Rel : A \to B$. The set
$Pre(Rel) := \{a \in A \mid \exists\, b \in B \; with \; (a, b) \in Rel\}$
is called the *pre-image* of the relation. The set
$Im(Rel) := \{b \in B \mid \exists\, a \in A \; with \; (a, b) \in Rel\}$
is called the *image* of the relation. For a relation Rel from A to B and a subset A' of A we set
$Rel(A') := \{b \in B \mid \exists\, a \in A' \; with \; (a, b) \in Rel\}$

R1.2 Definition: For a relation Rel from A to B the *inverse relation* (or transposed relation) Rel^T from B to A is given by
$Rel^T := \{(b, a) \in B \times A \mid (a, b) \in Rel\}$,
i.e we just change the order of the entries in the pairs belonging to Rel. Obviously $Pre(Rel^T) = Im(Rel)$, $Im(Rel^T) = Pre(Rel)$ and $Rel = (Rel^T)^T$.

R1.3 Definition: For a relation Rel from A to B and a relation Rel' from B to D the *composition* of Rel with Rel' is given by the relation
$Rel' \circ Rel := \{(a, d) \in A \times D \mid \exists\, b \in B \; with \; (a, b) \in Rel \; and \; (b, d) \in Rel'\}$
from A to D. Obviously
$Pre(Rel' \circ Rel) = Rel^T(Im(Rel) \cap Pre(Rel'))$,
$Im(Rel' \circ Rel) = Rel'(Im(Rel) \cap Pre(Rel'))$ and
$(Rel' \circ Rel)^T = Rel^T \circ (Rel')^T$.
In particular the composition may be the empty set.

2 Special Relations

R2.1 Definition: A relation Rel from A to B is called
surjective on A, if $A = Pre(Rel)$,

surjective on B, if $B = Im(Rel)$,
injective on B, if for any two pairs $(x_1, y_1), (x_2, y_2) \in Rel$ $x_1 = x_2$ implies $y_1 = y_2$.
injective on A, if for any two pairs $(x_1, y_1), (x_2, y_2) \in Rel$ $y_1 = y_2$ implies $x_1 = x_2$.

R2.2 Definition: For a relation Rel from A to B the *induced self-relations* Rel_A from A to A resp. $(Rel^T)_B$ from B to B are given by
$Rel_A := Rel^T \circ Rel$ resp.
$(Rel^T)_B := Rel \circ Rel^T$. Defining
$\Delta_X := \{(x, x) \mid x \in X\}$ for any set X (the *diagonal* of $X \times X$) we conclude
$\Delta_{Pre(Rel)} \subseteq Rel_A$,
$\Delta_{Pre(Rel^T)} = \Delta_{Im(Rel)} \subseteq (Rel^T)_B$,
$D \subseteq Rel_A(D)$ for all $D \subseteq Pre(Rel)$,
$D \subseteq Rel^T_B(D)$ for all $D \subseteq Pre(Rel^T) = Im(Rel)$.

R2.3 Definition: i) A relation Map from A to B is called a *map* from A to B if it is surjective on A (gobally defined) and injective on B (one-valued), i.e., $Pre(Map) = A$ and $Map(\{a_1\}) \neq Map(\{a_2\})$ implies $a_1 \neq a_2$.
ii) For a map Map from A to B we observe the following:
$Map_A \circ Map_A = Map_A$,
$Map \circ Map_A = Map$,
$(Map^T)_B = \Delta_{Im(Map)}$
$\{Map^T(\{b\}) \mid b \in Im(Map)\}$ provides a pairwise disjoint covering of A by subsets, having the same image element in B under Map.
iii) For a map Map from A to B a subset $Map' \subseteq Map$ is a map from $Pre(Map')$ to B, called the *restriction* of Map to $Pre(Map')$. The injectivity of Map implies that $Map' = \{(a, Map(\{a\})) \mid a \in Pre(Map')\}$, i.e.,
Map' is determined by $Pre(Map')$ and Map only. Viceversa, we may choose any non-empty subset $A' \subseteq A$ and get the(!) restriction $Map|A'$ of Map to A' by $Map|A' = \{(a, Map(a)) \mid a \in A')\}$, where the set brackets are omitted for the elements describing one-point sets.

R2.4 Definition: i) A map Map from A to B is called *surjective*, if it is surjective on B.
A map Map from A to B is called *injective*, if it is injective on A.
A map Map from A to B is called a *bijective*, if it is surjective and injective, i.e. the relation Map is surjective on A and B and it is injective on A and B.
ii) For a surjective map $Mapsur$ from A to B we observe the following:
By selecting for each $y \in B$ exactly one $x \in Mapsur^T(\{y\})$ we define a relation $Mapsur'$ fom B to A which is a map from B to A, i.e. surjective on B and injective on A, which is still injective on B and satisfies $Mapsur \circ Mapsur' = \Delta_B$. $Mapsur'$ is bijective from B to $Im(Mapsur')$.
iii) For an injective map $Mapin$ from A to B we observe the following:
$Mapin^T \circ Mapin = \Delta_A$. $Mapin$ is bijective from A to $Im(Mapin)$.
iv)For a bijective map $Mapbi$ from A to B we observe the following:
$Mapbi^T \circ Mapbi = \Delta_A$.

$Mapbi \circ Mapbi^T = \Delta_B$.
v) Restrictions of maps preserve injectivity, but not necessarily surjectivity.

R2.5 Definition: A relation Equ from A to A is called an *equivalence relation* on A, if it satisfies the following conditions:
i) $\Delta_A \subseteq Equ$ (*reflexivity*), i.e. each element of A is equivalent to itself.
ii) $Equ = Equ^T$ (*symmetry*), i.e. if a_1 is equivalent to a_2, then a_2 is equivalent to a_1.
iii) $Equ \circ Equ \subseteq Equ$ (*transitivity*), i.e. if a_1 is equivalent to a_2 and a_2 is equivalent to a_3, then a_1 is equivalent to a_3.
Remarks: For an equivalence relation Equ from A to A and an element $a \in A$ the set $[a] := \{b \in A \mid (a, b) \in Equ\}$ is called the *equivalence class* represented by a. It is easy to see that
i) $a \in [a]$ for all $a \in A$ by reflexivity,
ii) $a \in [b]$ implies $b \in [a]$ by symmetry, and
iii) $c \in [a] \cap [b]$ implies $[c] = [a] = [b]$ by transitivity.
This may be summarized as follows: A is covered by mutually disjoint equivalence classes of Equ. The set of these classes is called the *quotient* of A by Equ or better *factor set* of A by Equ.
Example from "nature" or primary school: The positive rational numbers are given by equivalence classes of pairs of positive integers (a, b), where
$((a, b), (c, d)) \in Equ$ if and only if $a \cdot d = b \cdot c$.
Everybody remembers the many exercises set up just for navigating in the equivalence class representing the same rational number.

R2.6 Definition: A relation Ord from A to A is called a *partial order* on A, if it satisfies the following conditions:
i) $\Delta_A \subseteq Ord$ (*reflexivity*),
ii) $Ord \circ Ord \subseteq Ord$ (*transitivity*),
iii) $(a, b) \in Ord$ and $(b, a) \in Ord$ implies $a = b$ (*antisymmetry*).
The definition of a partial order is refined to that of a *total order* on A by the following additional requirement:
iv) For any two elements $a, b \in A$ we have $(a, b) \in Ord$ or $(b, a) \in Ord$.
The main features of orders are maximal and minimal elements. Their definition is obvious.

3 Constructions

R3.1 Definition: For a relation Rel from A to B a relation $Rel' \subseteq Rel$ is called a *restriction* of Rel.
Obviously $Pre(Rel') \subseteq Pre(Rel)$, $Im(Rel') \subseteq Im(Rel)$,
$Rel'(A') \subseteq Rel(A')$ for every subset A' of $Pre(Rel')$.
For a map Map from A to B and a subset A' of A the restriction of Map given by
$Map \mid A' := \{(a, b) \in Map \mid a \in A'\}$
is a map again, called the restriction $Map \mid A'$ of Map to A'.

R3.2 Components: Let Rel_A be the induced self-relation from A to A for a relation Rel from A to B. Then it is easily seen that Rel_A is reflexive and symmetric. Considering all powers $(Rel_A)^h$ for $h \in \mathbb{N}$ we get an increasing sequence $(Rel_A)^h(\{a\})$ of subsets of $Pre(Rel) = Pre(Rel_A)$ for each $a \in Pre(Rel)$. Their union
$PathC(a) := \bigcup\{(Rel_A)^h(\{a\}) \mid h \in \mathbb{N}\}$
is called the *path component* of a with respect to Rel_A in $Pre(Rel)$.
If we consider the elements in $Pre(Rel)$ as the vertices of a graph, where two vertices are connected by an edge, if and only if they are related to the same element in B, then the path components of this graph are exactly the path components of Rel_A.
Belonging to the same path component is an equivalence relation from $Pre(Rel)$ to $Pre(Rel)$. Hence $Pre(Rel)$ is covered by the path components of Rel_A such that the covering sets are mutually disjoint.
If $Pre(Rel)$ is finite, then for each a in $Pre(Rel)$ there exists a minimal $h_a \in \mathbb{N}$ such that
$(Rel_A)^{h_a}(\{a\}) = PathC(a)$.
In particular we have $(Rel_A)^{h_a}(\{a\}) = (Rel_A)^{h_a+1}(\{a\})$. Furthermore, in the finite case the powers $(Rel_A)^h$ start to stabilize after a sufficiently high exponent.

4 Proofs, alternatives, further considerations

R4.1 Injectivity: Let Rel be a relation from A to B. The following properties are equivalent for the injectivity on A:
I: For any two pairs $(x_1, y_1), (x_2, y_2) \in Rel$ $y_1 = y_2$ implies $x_1 = x_2$.
II: For any two pairs $(x_1, y_1), (x_2, y_2) \in Rel$ $x_1 \neq x_2$ implies $y_1 \neq y_2$.
Proof of I implies II: Assume that for any two pairs $(x_1, y_1), (x_2, y_2) \in Rel$ $y_1 = y_2$ implies $x_1 = x_2$. Let there be $(w_1, z_1), (w_2, z_2) \in Rel$ such that $w_1 \neq w_2$. Assuming $z_1 = z_2$ leads with our initial assumption to $w_1 = w_2$. Hence this assumption leads to a contradiction, which implies $z_1 \neq z_2$.
The proof that II implies I is seen by the same kind of simple contradiction.

R4.2 Injectivity: The injectivity of Rel on A is equivalent to the following: For every $b \in Im(Rel)$ the set $Rel^T(b)$ has exactly one element.
The injectivity of Rel on B is equivalent to the following: For every $a \in Pre(Rel)$ the set $Rel(a)$ has exactly one element.
The proof is easy like that one for R4.1.

R4.3 Proposition: Let Rel be a relation from A to B. The following properties are equivalent:
I: $(a_1, a_2) \in Rel_A$.
II: There exists $b \in B$ with $(a_1, b), (a_2, b) \in Rel$.
III: $Rel(a_1) \cap Rel(a_2) \neq \emptyset$.
Proof: I implies II follows from the definition of the composition: $Rel_A := Rel^T \circ Rel$ gives for $(a_1, a_2) \in Rel_A$ that there exists $b \in B$ with $(a_1, b) \in Rel$ and

$(b, a_2) \in Rel^T$. The latter implies $(a_2, b) \in Rel$, which proves II. The proofs for the other implications are also very simple.

R4.4 Propositions: Let Rel be a relation from A to B. Then the following assertions are valid:
i) Rel_A is reflexive on $Pre(Rel)$.
ii) $(Rel_A)^h$ is reflexive on $Pre(Rel)$ for all natural numbers h.
iii) For every $D \subseteq Pre(Rel)$ we have $D \subseteq Rel_A(D)$.
iv) Rel_A is symmetric on $Pre(Rel)$.
v) $(Rel_A)^h$ is symmetric on $Pre(Rel)$ for all natural numbers h.
Proofs:
i) Let $a \in Pre(Rel)$. Then there exists $b \in Im(Rel)$ such that $(a, b) \in Rel$. Hence and $(b, a) \in Rel^T$, which implies $(a, a) \in Rel_A$ by R4.3.
ii) is implied by the fact that reflexivity is preserved under composition.
iii) is a simple implication from i).
iv) follows from R4.3, II.
v) is seen with iv) from the equation $((Rel_A)^2)^T = (Rel_A)^T \circ (Rel_A)^T = Rel_A \circ Rel_A = (Rel_A)^2$ and applying induction on h.

R4.5 Propositions: Let Rel be a relation from A to B. Let
$PathC(a) := \bigcup\{(Rel_A)^h(\{a\}) \mid h \in \mathbb{N}\}, a \in Pre(Rel)$,
be the path components of Rel_A as in R3.2. Then the following assertions are valid:
i) $a \in (Rel_A)^h(\{a\}) \subseteq (Rel_A)^{h+1}(\{a\})$ for all $a \in Pre(Rel)$ and all $h \in \mathbb{N}$. Hence $a \in PathC(a)$ for all $a \in Pre(Rel)$.
ii) $Rel_A(PathC(a)) = PathC(a)$ for all $a \in Pre(Rel)$.
iii) $a' \in PathC(a)$ implies $PathC(a) = PathC(a')$ for all $a, a' \in Pre(Rel)$.
iv) $PathC(a) \cap PathC(a') \neq \emptyset$ implies $PathC(a) = PathC(a')$.
Proofs:
i) follows from R4.4, part iii).
ii) $Rel_A(PathC(a)) = Rel_A(\bigcup\{(Rel_A)^h(\{a\}) \mid h \in \mathbb{N}\}) =$
$\{(Rel_A)^{h+1}(\{a\}) \mid h \in \mathbb{N}\} = \{(Rel_A)^s(\{a\}) \mid s \in \mathbb{N}\} = PathC(a)$
according to part i).
iii) I: $a' \in PathC(a)$ implies with R4.4, part iii), and R4.5, part ii),
$(Rel_A)^h(a') \subseteq (Rel_A)^h(PathC(a)) = PathC(a)$ for all $h \in \mathbb{N}$. This implies $PathC(a') \subseteq PathC(a)$
II: For $a' \in PathC(a)$ there exists by definition an $h_0 \in \mathbb{N}$ such that $a' \in (Rel_A)^{h_0}(\{a\})$, i.e., $(a, a') \in (Rel_A)^{h_0}$. Symmetry (see R4.4, part v)) implies $(a', a) \in (Rel_A)^{h_0}$. Hence $a \in (Rel_A)^{h_0}(\{a'\}$, leading to $a \in PathC(a')$, from which we conclude like in part I $PathC(a) \subseteq PathC(a')$.
The inclusions from part I and II give the proposed equality.
iv) This follows from part iii), because any element in the intersection will lead to a component coinciding with both, $PathC(a)$ and $PathC(a')$, which implies the proposed equality.

R4.6 Remark: The assertions from R4.5 imply, that belonging to the same path component with respect to Rel_A is an equivalence relation. The path components are the equivalence classes. They provide a disjoint covering of $Pre(Rel)$

R4.7 Definition: Let Rel be a relation from A to B. A subset D of $Pre(Rel)$ is called Rel_A *invariant*, if $Rel_A(D) = D$.

R4.8 Proposition: The minimal (with respect to inclusion) Rel_A-invariant sets are exactly the path components of Rel_A.
Proof: I: Let D be a minimal Rel_A-invariant set. Let $a \in D$. Then $(Rel_A)^h(a) \subseteq (Rel_A)^h(D) = D$ for all $h \in \mathbb{N}$ by Rel_A-invariance of D, and hence $PathC(a) \subseteq D$. Minimality of D implies $PathC(a) = D$.
II: That pathcomponents are Rel_A-invariant follows from the above Propositions. Remains to show that they are minimal with respect to this property. Let D be a Rel_A-invariant set with $D \subseteq PathC(a)$. Then for $a' \in D$ we get by the Rel_A-invariance of D $(Rel_A)^h(a') \subseteq (Rel_A)^h(D) = D$ for all $h \in \mathbb{N}$ and hence, because $a' \in D \subseteq PathC(a)$,
$PathC(a) = PathC(a') \subseteq D \subseteq PathC(a)$,
where the first equality follows from R4.6, part iii). This implies the equality also for the inclusions, i.e., $D = PathC(a)$. Hence $PathC(a)$ has no proper Rel_A-invariant subset.

R4.9 Propositions: i) Let D be Rel_A-invariant. Then $Rel(D)$ is Rel^T_B-invariant.
ii) Let E be Rel^T_B-invariant. Then $Rel^T(E)$ is Rel_A-invariant.
iii) Let $PathC(a)$ be a path component with respect to Rel_A. Then $Rel(PathC(a))$ is a path component with respect to Rel^T_B and $Rel(PathC(a)) = PathC^T(b)$ for all $b \in Rel(a)$, $PathC^T(b)$ denoting the path component of b with respect to Rel^T_B.
iv) Let $PathC^T(b)$ be a path component with respect to Rel^T_B. Then $Rel^T(PathC^T(b))$ is a path component with respect to Rel_A and $Rel^T(PathC^T(b)) = PathC(a)$ for all $a \in Rel^T(b)$.
v) Let $PCRel_A$ be the set of path components in A with respect to Rel_A, $PCRel^T_B$ be the set of path components in B with respect to Rel^T_B. Then $RelComp$ defined by $RelComp(AComp) := Rel(AComp)$ for all $AComp \in PCRel_A$ (where $AComp$ on the left-hand side is considered as an element of the set $PCRel_A$ and $AComp$ on the right-hand side is considered as a subset of A, to which Rel is applied) is a bijective map from $PCRel_A$ to $PCRel^T_B$.
Proofs: i) Let D be Rel_A-invariant, i.e., $D = Rel^T(Rel(D))$. Then
$Rel^T_B(Rel(D)) = Rel(Rel^T(Rel(D))) = Rel(D)$.
ii) The proof is similar to the proof of part i).
iii) Let $b \in Rel(a)$. Then
I: $(Rel^T_B)^h(b) \subseteq (Rel^T_B)^h(Rel(a)) = Rel((Rel_A)^h(a))$ for all $h \in \mathbb{N}$. This implies $PathC^T(b) \subseteq Rel(PathC(a))$.
II: It also implies $a \in Rel^T(b)$. Hence
$(Rel_A)^h(a) \subseteq (Rel_A)^h(Rel^T(b)) = Rel^T((Rel^T_B)^h(b))$ for all $h \in \mathbb{N}$ and hence
$Rel((Rel_A)^h(a)) \subseteq Rel((Rel_A)^h(Rel^T(b))) = Rel(Rel^T((Rel^T_B)^h(b))) = (Rel^T_B)^{h+1}(b)$ for all $h \in \mathbb{N}$, giving $Rel(PathC(a)) \subseteq PathC^T(b)$. The inclusions from I and II give the

equality $Rel(PathC(a)) = PathC^T(b)$.
iv) The proof is similar to the proof of part iii).
v) From parts iii) and iv) we get that *RelComp* is a map and that
$Rel^T Comp(BComp) := Rel^T(BComp)$ for all $BComp \in PCRel_B^T$ is a map, which is inverse to *RelComp*. Hence *RelComp* and $Rel^T Comp$ are bijective maps.

R4.10 Decomposition Theorem: Let *Rel* be a relation from A to B. *Rel* decomposes into pairwise disjoint restrictions to $AComp \times RelComp(AComp)$, $AComp \in PCRel_A$, being surjective on both parts, having pairwise disjoint preimages *AComp* covering $Pre(Rel)$, and having pairwise disjoint images $RelComp(AComp)$ covering $Im(Rel)$:
$Rel = Rel \cap \bigcup\{AComp \times RelComp(AComp) \mid AComp \in PCRel_A\}$.
The same equality holds for $PCRel_B^T$:
$Rel = Rel \cap \bigcup\{Rel^T Comp(BComp) \times BComp \mid BComp \in PCRel_B^T\}$.
(For the notations see R4.9.)
Proof: The proof follows immediately from the propositions under R4.9, where the inclusion
$Rel \subseteq Rel \cap \bigcup\{AComp \times RelComp(AComp) \mid AComp \in PCRel_A\}$ is seen as follows:
$(a, b) \in Rel$ gives $a \in PathC(a)$ and $b \in Rel(a) \subseteq Rel(PathC(a))$. Hence
$(a, b) \in Rel \cap (PathC(a) \times Rel(PathC(a))) \subseteq$
$Rel \cap \bigcup\{AComp \times RelComp(AComp) \mid AComp \in PCRel_A\}$.

R4.11 Remarks: i) The decomposition in 4.10 can be considered as maximal in the following sense: There is no further refinement of the decomposition. The single parts $Rel \cap (AComp \times RelComp(AComp))$ are indecomposable. They have only one component in the preimage and one component in the image.
ii) If the relation is a map from A to B, then the theorem is obvious. The components of *map* in the image set consist of single points $b \in B$, which are values of *map*, the components in the preimage are the classical (complete) preimages $map^T(b)$ of these image points. The decomposition theorem states, that *map* can be maximally decomposed into a set of restrictions, such that no two of these restrictions have a common value. These restrictions have to be constant maps then, having the mutually disjoint complete preimages of the unique values as their range of definition.
iii) The considerations above are quite elementary and developed in a manner, which does not need further background. They are likely to be found in the current or in another way in textbooks on discrete mathematics, relations, graphs, in particular bipartite graphs or combinatorics.
For example, a relation *Rel* may be considered as a bipartite graph with A and B as sets of vertices, the relation marking the vertices being connected by edges. The sets A and B are considered as disjoint, and at the initial setting there are no edges inside A or B. These are developed as derived items by Rel_A and Rel_B^T. The decomposition theorem separates the bipartite graph into a maximal set of mutually disconnected (or combinatorically independent) bipartite subgraphs, which are indecomposable then. In the case of a map the graph is very simple.

iv) Several assertions made for Rel_A immediately transfer to general self-relations from A to A.

No proofs: There will be no proofs for the further considerations though they can be given on special request.

R4.12 Definition and Remarks: Let Rel be a relation from the finite set A to the finite set B. For sake of simplicity we assume that Rel is surjective on A and B.
i) In this case for every $a \in A$ there is a uniquely determined minimal $h(a) \in \mathbb{N}$ (0 included) such that $(Rel_A)^{h(a)}(a) = PathC(a)$. $h(a)$ is called the *order* of a with respect to Rel_A. The *order* $h(b)$ of $b \in B$ with respect to Rel_B^T is defined in the same way.
ii) The order with respect to Rel_A may be different for two elements in the same component. For $(a, b) \in Rel$ $h(a)$ with respect to Rel_A and $h(b)$ with respect to Rel_B^T may be different.
iii) For each $AComp \in PCRel_A$ its *diameter* $Diam(AComp)$ with respect to Rel_A is defined by
$Diam(AComp) \;=\; \max\{h(a) \mid a \in AComp\}$.
$Diam(BComp)$ with respect to Rel_B^T is defined in the same way.

R4.13 Propositions: i) $Diam(AComp) = 0$ for all $AComp \in PCRel_A$, if and only if Rel is injective on A.
ii) $Diam(BComp) = 0$ for all $BComp \in PCRel_B^T$, if and only if Rel is injective on B.

R4.14 Propositions: i) $Diam(AComp) \leq 1$ for all $AComp \in PCRel_A$, if and only if Rel_A is transitive.
ii) $Diam(BComp) \leq 1$ for all $BComp \in PCRel_B^T$, if and only if Rel_B^T is transitive.

R4.15 Proposition: $|Diam(AComp) - Diam(RelComp(AComp))| \leq 1$, $|Diam(BComp) - Diam(Rel^T Comp(BComp))| \leq 1$.

R4.16 Definition: A *chain* with respect to Rel_A in A is given by a finite number of pairs $(a_\lambda, a_{\lambda+1}) \in Rel_A$, $\lambda = 1, \ldots, \Lambda$, with the additional requirement, that there are no trivial parts in the chain, i.e., $a_\lambda \neq a_{\lambda+1}$ for all $\lambda = 1, \ldots, \Lambda$. Λ is called the *length* of the chain, a_1 is called its *initial point* and $a_{\Lambda+1}$ its *end point*. The chain is *connecting* a_1 with $a_{\Lambda+1}$ with respect to Rel_A.

R4.17 Proposition: Let $(a_\lambda, a_{\lambda+1}) \in Rel_A$, $\lambda = 1, \ldots, \Lambda$, be a chain with respect to Rel_A and assume that $a_{\lambda+1} \in (Rel_A)^\lambda(a_1) \setminus Rel_A^{\lambda-1}(a_1)$ for all $\lambda = 1, \ldots, \Lambda$. Then the length of the chain is minimal among all chains connecting a_1 with $a_{\Lambda+1}$.

R4.18 Remark: R4.17 is the starting point for constructing minimal spanning trees composed of chains of minimal lengths in $AComp$ with a given $a \in AComp$ as initial point.